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On the harmonic oscillations of a rigid body on a free surface

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The present paper deals with the practical and rigorous solution of the potential problem associated with the harmonic oscillation of a rigid body on a free surface. The body is assumed to have the form of either an elliptical cylinder or an ellipsoid. The use of Green's function reduces the determination of the potential to the solution of an integral equation. The integral equation is solved numerically and the dependency of the hydrodynamic quantities such as added mass, added moment of inertia and damping coefficients of the rigid body on the frequency of the oscillation is established.

1. Introduction

A fluid motion caused by small prescribed oscillations of a rigid body on the free surface of an incompressible inviscid fluid is studied in this paper. The fluid is assumed to occupy a space bounded by the surface of the body and by the free surface extending in all directions. The induced motion of the fluid in this space interacts with the oscillating body and exerts the dynamic pressure on the immersed surface. A point of interest here is to find the effects of such pressure on the variation of the inertial and damping characteristics of the body performing a steady oscillation with a certain frequency.

In the formulation of the present problem, the boundary conditions will be linearized by neglecting high-order terms in view of the smallness of the motions involved. It is well known then that the potential which satisfies the linear conditions on the boundaries in the undisturbed position and the proper physical condition at infinity can be determined uniquely. Nevertheless, such a potential depends on the mode and frequency of the oscillation as well as the form of the body.

For a body of general form Kotchin (1940) and John (1950) showed that the solution of the problem can be represented as a potential corresponding to a surface distribution of point sources. The strength of the sources is to be determined from a Fredholm integral equation which satisfies a prescribed kinematic condition (appropriate to a specific oscillation) for the potential on the body surface.

We present a procedure for the approximate solution of these integral equations. In two-dimensional problems the kernel of the integral equation is continuous, but in three-dimensional problems it becomes singular; the handling of the singular term is one of the major problems of the present work. This procedure is valid for any shape which can be described analytically.

Once the equations are solved, the dynamic forces $\mathbf{F}_d$ and moments $\mathbf{G}_d$ on the rigid body can be obtained by numerical quadrature. For instance, in three-dimensional problems resolving these into a component in phase with the acceleration and the other component in phase with the velocity, we write

$$\mathbf{F}_d = -\rho\bar{a}^3\mathbf{M}\ddot{\bar{X}}_j - \rho\sigma\bar{a}^3\mathbf{N}\dot{\bar{X}}_j \quad (j = 1, 2, 3)$$

for the linear oscillations $\bar{X}_j(t) = \mathrm{Re}\,[\bar{X}_j^0(t)\, e^{-i\sigma t}]$, and

$$\mathbf{G}_d = -\rho\bar{a}^4\mathbf{I}\ddot{\theta}_j - \rho\sigma\bar{a}^4\mathbf{H}\dot{\theta}_j \quad (j = 4, 5, 6)$$

for the angular oscillations

$$\theta_j(t) = \mathrm{Re}\,[\theta_j^0(t)\, e^{-i\sigma t}],$$

where ρ represents the density of the fluid, $\bar{a}$ is the half-length of the body, and σ is the frequency of oscillation. The dimensionless quantities $\mathbf{M}$ and $\mathbf{N}$ are called the added mass and linear damping coefficient, and similar quantities $\mathbf{I}$ and $\mathbf{H}$ are called the added moment of inertia and angular damping coefficient, respectively. For the case of two-dimensional problems the dynamic forces $\mathbf{F}_d^{(2)}$ and moments $\mathbf{G}_d^{(2)}$ per unit length can be written in the same form with $\bar{a}^3$ and $\bar{a}^4$ replaced by $\bar{a}^2$ and $\bar{a}^3$, respectively.

These hydrodynamic quantities are functions of frequency or, more precisely, functions of the parameter $\bar{a}\sigma^2/g = a$, g being the acceleration of gravity. In this paper we present results for simple shapes, ellipses in two dimensions and ellipsoids in three dimensions. In order to gain some insight into the validity of the present method, the results are compared with the previous results obtained by a quite different method which was originally developed by Ursell (1949*a*, *b*). Added mass and the damping coefficient of heaving ellipses presented in figures 15 and 16 showed good agreement with the results of Ursell (1949*a*) and Porter (1960). The same agreement is noted between the present results of surging ellipses in figures 13 and 14 and the work of Tasai (1961) who used Ursell's method. The only results in three dimensions which are known to us are due to Havelock (1955) and Barakat (1962) and pertain only to a heaving sphere. These are compared with the corresponding curves in figures 3 and 4. It was noted that the present results are in complete agreement with those of Havelock. It would seem, on the basis of the limited evidence available, that the present procedure yields accurate results.

Nevertheless, there is a fundamental limitation to the present numerical method. The kernels of the integral equations oscillate rapidly as the parameter a increases. Therefore, unless many subdivisions are used, the numerical quadratures occurring in the present procedure become inaccurate. In this paper, we have limited the computations to values of a less than four, which covers the range of practical interest. For instance, in the three dimensions, it means that we have waves which are not much shorter than the length of the rigid body. Higher frequency results can be obtained with more computational labour.

The final objective of the present study is to assess the merit of various models adopted for the investigation of ship form. At first, the hydrodynamic quantities such as the added mass, added moment of inertia, and damping coefficients of the ellipsoids having different axis ratios are estimated by the strip theory, using the two-dimensional data of a long cylinder. Next, the effect of the draft on the ellipsoids is examined in order to evaluate the applicability of the shallow-draft approximation (see Kim 1963).

2. General formulation

Consider a rigid body immersed in an inviscid, incompressible fluid with its centre of gravity on the origin O, and its axes on the $(\bar{x}, \bar{z})$-plane of a space co-ordinate system $O\bar{x}\bar{y}\bar{z}$. The $(\bar{x}, \bar{z})$-plane here coincides with the undisturbed free surface, and we take the $\bar{y}$-axis positive upwards. If the body is given linear and angular oscillations of small amplitudes $\bar{\mathbf{X}}^0$ and $\mathbf{\Theta}^0$ with a certain frequency σ about its equilibrium position, viz.

$$\left.\begin{aligned} \bar{\mathbf{X}}(t) &= \mathrm{Re}\,[\bar{\mathbf{X}}^0 e^{-i\sigma t}], \\ \mathbf{\Theta}(t) &= \mathrm{Re}\,[\mathbf{\Theta}^0 e^{-i\sigma t}], \end{aligned}\right\} \tag{2.1}$$

the surface disturbance created by these motions travel outwards as waves in all directions.

The position of the immersed surface $S(\bar{x}, \bar{z})$ relative to the space co-ordinates at any instance can now be expressed by specifying a position vector of the centre of gravity $\bar{\mathbf{X}} = \hat{\mathbf{i}}\bar{X}_1 + \hat{\mathbf{j}}\bar{X}_2 + \hat{\mathbf{k}}\bar{X}_3$, and the Eulerian angle $\mathbf{\Theta} = \hat{\mathbf{i}}\theta_4 + \hat{\mathbf{j}}\theta_5 + \hat{\mathbf{k}}\theta_6$. The linear components $\bar{X}_1$, $\bar{X}_2$ and $\bar{X}_3$ are called surge, heave and sway, while the angular components θ_4, θ_5 and θ_6 are named roll, yaw and pitch, respectively. Assuming the fluid to attain a time-periodic irrotational motion when transient motion dissipates, a velocity potential

$$\Phi(\bar{x}, \bar{y}, \bar{z}; t) = \mathrm{Re}\,[V(\bar{x}, \bar{y}, \bar{z})\, e^{-i\sigma t}] \tag{2.2}$$

may be introduced to describe the state of the fluid. The incompressibility then requires that V is a solution of the potential equation

$$\nabla^2 V(\bar{x}, \bar{y}, \bar{z}) = 0 \quad \text{in} \quad \bar{y} < 0, \tag{2.3}$$

where V is complex-valued.

As the amplitude of oscillation is considered small, the amplitude of induced wave motion will also be small in comparison with the wavelength. Therefore, the linearized dynamic condition for the velocity potential at the undisturbed free surface becomes

$$g\,\bar{\eta}(\bar{x}, \bar{z}; t) + \Phi_t(\bar{x}, 0, \bar{z}; t) = 0, \tag{2.4}$$

where $\bar{\eta}$ represents the free surface elevation, and g the acceleration of gravity. Then the linearized kinematic condition $\Phi_{\bar{y}}(\bar{x}, 0, \bar{z}; t) = \bar{\eta}_t(\bar{x}, \bar{z}; t)$ and (2.4) yields

$$\frac{\partial}{\partial \bar{y}} V(\bar{x}, 0, \bar{z}) - kV(\bar{x}, 0, \bar{z}) = 0 \quad \text{on} \quad \bar{y} = 0, \tag{2.5}$$

with k denoting the wave-number which is equal to $\sigma^2/g = 2\pi/\bar{\lambda}$, $\bar{\lambda}$ being the length of free wave.

The kinematic condition which states that in the absence of viscosity the normal velocity across the immersed surface of the body is continuous takes the form

$$\frac{\partial}{\partial \bar{n}} \Phi(\bar{x},\bar{y},\bar{z};t) = (\dot{\mathbf{X}}+\dot{\mathbf{\Theta}}\times\bar{\mathbf{r}}).\mathbf{n}, \tag{2.6}$$

where $\bar{\mathbf{r}}$ represents the position vector of a material point on the body and $\mathbf{n}$, the unit normal of the immersed surface, i.e.

$$\bar{\mathbf{r}} = \hat{\mathbf{i}}\bar{x}+\hat{\mathbf{j}}\bar{y}+\hat{\mathbf{k}}\bar{z} \quad \text{and} \quad \mathbf{n} = \hat{\mathbf{i}}n_x+\hat{\mathbf{j}}n_y+\hat{\mathbf{k}}n_z.$$

Note that as the consequence of the linearization, the kinematic condition is to be satisfied on the surface in the undisturbed position. Thus, we find

$$\frac{\partial}{\partial \bar{n}} V(\bar{x},\bar{y},\bar{z}) = \sum_{j=1}^{6} \frac{\partial}{\partial \bar{n}} V_j(\bar{x},\bar{y},\bar{z}) = -i\sigma[\bar{\mathbf{X}}^0.\mathbf{n}+\mathbf{\Theta}^0.(\bar{\mathbf{r}}\times\mathbf{n})], \tag{2.7}$$

for six degrees of freedom in the problem.

Finally, a disturbance in the finite region should produce only an outgoing progressive wave at large distance,

$$V(\bar{r},\theta,\bar{y}) - A(\theta)\,\bar{r}^{-\frac{1}{2}}\,e^{k\bar{y}+ik\bar{r}} \to 0 \quad \text{as} \quad \bar{r}\to\infty, \tag{2.8}$$

where $\bar{r} = (\bar{x}^2+\bar{z}^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(\bar{z}/\bar{x})$.

In order to show clearly the dependence of the solution on the frequency parameter $a = \bar{a}k$, $\bar{a}$ being a typical length of the body (such as a half-length of the ellipsoid or a half-beam of the cylinder), we shall first make the space variables and the amplitudes dimensionless, i.e.

$$x = \bar{x}/\bar{a}, \quad y = \bar{y}/\bar{a}, \quad z = \bar{z}/\bar{a} \quad \text{and} \quad X^0 = \bar{X}^0/\bar{a},$$

then introduce the pressure function u_j by

$$\left.\begin{aligned} i\sigma V_j(\bar{x},\bar{y},\bar{z})/g\bar{a}X_j^0 &= au_j(x,y,z) \quad (j = 1,2,3),\\ i\sigma V_j(\bar{x},\bar{y},\bar{z})/g\bar{a}\theta_j^0 &= au_j(x,y,z) \quad (j = 4,5,6).\end{aligned}\right\} \tag{2.9}$$

It follows that the dynamic pressure Π_j can now be expressed as

$$\left.\begin{aligned} \Pi_j(\bar{x},\bar{y},\bar{z};t) &= -\rho[\Phi_j(\bar{x},\bar{y},\bar{z};t)]_t = \rho g\bar{a}\,\mathrm{Re}\,[X_j^0 au_j(x,y,z)] \quad (j = 1,2,3),\\ \Pi_j(\bar{x},\bar{y},\bar{z};t) &= -\rho[\Phi_j(\bar{x},\bar{y},\bar{z};t)]_t = \rho g\bar{a}\,\mathrm{Re}\,[\theta_j^0 au_j(x,y,z)] \quad (j = 4,5,6).\end{aligned}\right\} \tag{2.10}$$

The boundary value problem which arises in the study of small harmonic oscillations of a body on the free surface is to find a potential $u_j(x,y,z)$, $j = 1,2,3,\ldots,6$, continuous in the fluid space such that

$$\left.\begin{aligned} &\text{(A)} \quad \nabla^2 u_j(x,y,z) = 0 \quad \text{in} \quad y<0,\\ &\text{(B)} \quad \frac{\partial}{\partial y} u_j(x,0,z) - au_j(x,0,z) = 0 \quad \text{outside } S(x,z),\\ &\text{(C)} \quad \frac{\partial}{\partial n} u_j(x,y,z) = h_j(x,y,z) \quad \text{on} \quad S(x,z),\\ &\text{(D)} \quad u_j(r,\theta,y) - A_j(\theta)\,r^{-\frac{1}{2}}\,e^{ay+iar} \to 0 \quad \text{as} \quad r\to\infty,\end{aligned}\right\} \tag{2.11}$$

where $S(x,z)$ represents the immersed surface of the body in the undisturbed position $S(x,y) = \bar{a}^{-1}\bar{S}(\bar{a}x, \bar{a}y)$, and h_j denotes the prescribed function which depends on the mode of oscillation. For six degrees of freedom, the functions h_j are given, respectively, by

$$\left.\begin{aligned} &h_1(x,y,z) = n_x, && h_2(x,y,z) = n_y, && h_3(x,y,z) = n_z, \\ &h_4(x,y,z) = yn_z - zn_y, && h_5(x,y,z) = zn_x - xn_z, && h_6(x,y,z) = xn_y - yn_x. \end{aligned}\right\} \quad (2.12)$$

In the case where a rigid body is a cylinder, we expect the solution of the problem to be two-dimensional; in effect, a solution for an $(\bar{x},\bar{y})$-plane which is normal to the axis of the cylinder is valid for all planes cutting the axis parallel to this plane. Therefore, the boundary value problem is to find a potential $u_j(x,y)$, $j = 1,2,3$, continuous in the fluid space such that

$$\left.\begin{aligned} &(a) \quad \nabla^2 u_j(x,y) = 0 \quad \text{in} \quad y < 0, \\ &(b) \quad \frac{\partial}{\partial y} u_j(x,0) - a u_j(x,0) = 0 \quad \text{outside } C(x), \\ &(c) \quad \frac{\partial}{\partial n} u_j(x,y) = h_j(x,y) \quad \text{on } C(x), \\ &(d) \quad u_j(x,y) - B_j e^{ay+iax} \to 0 \quad \text{as} \quad x \to \pm\infty, \end{aligned}\right\} \quad (2.13)$$

where $C(x)$ represents the immersed periphery of the cylinder in the undisturbed position; $C(x) = \bar{a}^{-1}\bar{C}(\bar{a}x)$ and h_j are given by

$$h_1(x,y) = n_x, \quad h_2(x,y) = n_y, \quad h_3(x,y) = xn_y - yn_x. \qquad (2.14)$$

Here we remark that due to the renaming of the co-ordinate axes $j = 1,2$, and 3 now correspond to the case of sway, heave and roll in the three-dimensional problem, respectively.

3. Representation of the potential

The source potentials G of unit strength in the lower half-space which satisfy the sets of boundary conditions (2.11 A, B, D) and (2.13 a,b,d) can be expressed in the form

$$\begin{aligned} G(x,y,z;\xi,\eta,\zeta) = R^{-1} + R'^{-1} - \pi a\, e^{a(y+\eta)}[S_0(a\varpi) + Y_0(a\varpi) - i2J_0(a\varpi)] \\ - 2a\, e^{a(y+\eta)} \int_{y+\eta}^{0} e^{-a\mu}(\mu^2 + \varpi^2)^{-\frac{1}{2}}\, d\mu, \end{aligned} \qquad (3.1)$$

$$\begin{aligned} G(x,y;\xi,\eta) = \ln\varpi + \ln\varpi' + 2\,e^{a(y+\eta)}[\cos a(x-\xi)\,\mathrm{Ci}\,a|x-\xi| \\ + \sin a|x-\xi|\,\mathrm{Si}\,a|x-\xi| - \ln|x-\xi| + \tfrac{1}{2}\pi\sin a|x-\xi| - i\pi\cos a(x-\xi)] \\ - 2a\,e^{a(y+\eta)} \int_{y+\eta}^{0} e^{-a\mu}\ln[(x-\xi)^2 + \mu^2]^{\frac{1}{2}}\, d\mu. \end{aligned} \qquad (3.2)$$

where

$$\varpi = [(x-\xi)^2 + (y-\eta)^2]^{\frac{1}{2}}, \quad \varpi = [(x-\xi)^2 + (y+\eta)^2]^{\frac{1}{2}}$$
$$R = [\varpi^2 + (z-\zeta)^2]^{\frac{1}{2}}, \qquad R' = [\varpi'^2 + (z-\zeta)^2]^{\frac{1}{2}},$$

and $S_0(a\varpi)$ is the Struve function of order 0, $J_0(a\varpi)$ and $Y_0(a\varpi)$ are the Bessel functions of the first and second kind of order zero, respectively, and $\mathrm{Si}\,a|x-\xi|$ and $\mathrm{Ci}\,a|x-\xi|$ denote the integral sine and cosine functions. It should be noted that the source potentials G are more tractable in the present form than in other expressions using Cauchy's principal-value integrals (see, for example, Wehausen & Laitone 1960).

We now seek the solution of the boundary value problems (2.11) and (2.13) in the following form:

$$u_j(x,y,z) = \frac{1}{4\pi}\iint_S f_j(\xi,\eta,\zeta)\,G(x,y,z;\xi,\eta,\zeta)\,dS, \tag{3.3}$$

$$u_j(x,y) = \frac{1}{2\pi}\iint_C f_j(\xi,\eta)\,G(x,y;\xi,\eta)\,dC, \tag{3.4}$$

where f represents the strength of distributed sources over the immersed surface $S(x,z)$ or $C(x)$, and is a continuous complex function.

According to potential theory, the normal derivatives of the potential u_j on $S(x,z)$ and $C(x)$ are given by

$$\frac{\partial}{\partial n}u_j(x,y,z) = -\tfrac{1}{2}f_j(x,y,z)+\frac{1}{4\pi}\iint_S f_j(\xi,\eta,\zeta)\frac{\partial}{\partial n}G(x,y,z;\xi,\eta,\zeta)\,dS, \tag{3.5}$$

and

$$\frac{\partial}{\partial n}u_j(x,y) = -\tfrac{1}{2}f_j(x,y)+\frac{1}{2\pi}\int_C f_j(\xi,\eta)\frac{\partial}{\partial n}G(x,y;\xi,\eta)\,dC. \tag{3.6}$$

Therefore, if f is determined from the integral equation

$$-f_j(x,y,z)+\frac{1}{2\pi}\iint_S f_j(\xi,\eta,\zeta)\frac{\partial G}{\partial n}\,dS = 2h_j(x,y,z), \tag{3.7}$$

or

$$-f_j(x,y)+\frac{1}{\pi}\int_C f_j(\zeta,\eta)\frac{\partial G}{\partial n}\,dC = 2h_j(x,y), \tag{3.8}$$

u_j will satisfy the boundary condition (2.11 C) or (2.13 c). Accordingly, u_j is the solution of the given problem. In order that these integral equations be soluble, the homogeneous equations must possess only the trivial solution. John (1950) proved that the homogeneous equation cannot have a non-trivial solution for sufficiently large wavelength.

We shall examine here the behaviour of the source potential and its normal derivative at the proximity of a point source: in (3.1) as a variable point (ξ,η,ζ) tends to the point source at (x,y,z) on the immersed surface $S(x,z)$, i.e. $R\to 0$, and $\varpi\to 0$, in addition to R^{-1} being singular, $Y_0(a\varpi)$ and $\int_{y+\eta}^{0} e^{-a\mu}(\mu^2+\varpi^2)^{-\frac{1}{2}}\,d\mu$ become logarithmically singular since

$$\left.\begin{aligned} &\lim_{\varpi\to 0}\left[Y_0(a\varpi)-\frac{2}{\pi}\ln(a\varpi)\right] = 0,\\ \text{and}\quad &\lim_{\varpi\to 0}\left[\int_{y+\eta}^{0} e^{-a\mu}(\mu^2+\varpi^2)^{-\frac{1}{2}}\,d\mu-\ln\frac{R'-y-\eta}{r}\right] = 0.\end{aligned}\right\} \tag{3.9}$$

However, because of the opposite signs these logarithmic terms cancel out and we obtain

$$G(x,y,z;\xi,\eta,\zeta) = R^{-1}+G^*(x,y,z;\xi,\eta,\zeta), \tag{3.10}$$

with

$$G^*(x,y,z;\xi,\eta,\zeta) = R'^{-1}-2a\,e^{a(y+\eta)}\Big\{\ln[a(R'-y-\eta)] + \tfrac{1}{2}\pi[S_0(a\varpi)+N_0(a\varpi)-i2J_0(a\varpi)]+\int_{y+\eta}^{0}(e^{-a\mu}-1)\,(\mu^2+\varpi^2)^{-\frac{1}{2}}\,d\mu\Big\},$$

where $N_0(a\varpi)$ is a regular function defined by

$$N_0(a\varpi) = Y_0(a\varpi) - \tfrac{1}{2}\pi \ln (a\varpi). \tag{3.11}$$

Next in (3.2) as (ξ, η) approaches the source at (x, y) along the periphery $C(x)$, i.e. $\varpi \to 0$, and $|x-\xi| \to 0$, in addition to $\ln \varpi$ and $\ln |x-\xi|$ being singular, $\mathrm{Ci}\, a|x-\xi|$ behaves as

$$\lim_{\xi \to x} [\mathrm{Ci}\, a|x-\xi| - (\gamma + \ln a|x-\xi|)] = 0, \tag{3.12}$$

where γ is Euler's constant. Here the opposite signs on the logarithmic terms again cancel singularities so that we have

$$G(x, y; \xi, \eta) = \ln \varpi + G^*(x, y; \xi, \eta), \tag{3.13}$$

with

$$G^*(x,y;\xi,\eta) = \ln \varpi' + 2\, e^{a(y+\eta)} \Big\{ \cos a(x-\xi)\, \mathrm{Ci}\, a|x-\xi| + \sin a|x-\xi|\, \mathrm{Si}\, a|x-\xi| - \ln|x-\xi| + \tfrac{1}{2}\pi \sin a|x-\xi| - i\pi \cos a(x-\xi) + a(y+\eta)\ln \varpi' + a(x-\xi)\tan^{-1}\frac{y+\eta}{x-\xi} - a(y+\eta) - \tfrac{1}{2}a \int_{y+\eta}^{0} (e^{-a\mu}-1)\ln[(x-\xi)^2+\mu^2]\, d\mu \Big\}.$$

Let us now turn to the normal derivatives of the source functions. From (3.10) and (3.13) we find that

$$\frac{\partial}{\partial n} G(x,y,z;\xi,\eta,\zeta) = \frac{\partial}{\partial n} R^{-1} + \frac{\partial}{\partial n} G^*(x,y,z;\xi,\eta,\zeta), \tag{3.14}$$

and

$$\frac{\partial}{\partial n} G(x,y;\xi,\eta) = \frac{\partial}{\partial n} \ln \varpi + \frac{\partial}{\partial n} G^*(x,y;\xi,\eta). \tag{3.15}$$

In (3.14) as R and ϖ tend to zero $(\partial/\partial n) R^{-1}$ becomes indeterminate and the terms in $(\partial/\partial n) G^*$ which contain powers of reciprocal distance such as ϖ^{-1} or ϖ^{-2} becomes singular. However, when $\varpi = 0$ the multiple factors of the reciprocal distances vanish simultaneously so that these reciprocal distance terms do not present the problem here (Kim 1964, Appendix). Finally in (3.15), as ϖ and $|x-\xi|$ tend to zero, $(\partial/\partial n)\ln \varpi$ takes the value of $(2R_0)^{-1}$, where R_0 is the radius of curvature of the periphery at $\xi = x$. Therefore, no singular term appears in (3.15).

Thus, the integrals containing the terms R^{-1}, $(\partial/\partial n) R^{-1}$ and $\ln \varpi$ in their integrands have to be evaluated using the special scheme.

4. Forces and moments

The forces and moments caused by the dynamic fluid pressure acting upon the immersed surface of an oscillating rigid body may be resolved into components in phase with the acceleration and other components in phase with the velocity of the rigid body. The former quantities are called added mass or added moment of inertia while the latter quantities are called linear or angular damping coefficients.

We shall determine the forces and moments when an ellipsoid is excited into small harmonic oscillations about it equilibrium position on a free surface of a fluid. Since the form of an ellipsoid is symmetric about the space axes lying on the free surface, from the result presented by John (1949), the forces $\mathbf{F}$ and moments $\mathbf{G}$ can be deduced as

$$\left.\begin{aligned} \mathbf{F}(t) &= \mathbf{F}_d(t) - \hat{\mathbf{j}}\rho g \bar{X}_2(t)\, \pi \bar{a}\bar{b} \quad (j = 1,2,3), \\ \text{and} \quad \mathbf{G}(t) &= \mathbf{G}_d(t) - \hat{\mathbf{i}}\rho g \theta_4(t)\, \tfrac{1}{4}\pi \bar{a}\bar{b}(\bar{b}^2+\bar{c}^2) - \hat{\mathbf{k}}\rho g \theta_6(t)\, \tfrac{1}{4}\pi \bar{a}\bar{b}(\bar{a}^2+\bar{c}^2) \quad (j = 4,5,6), \end{aligned}\right\} \tag{4.1}$$

where the dynamic forces and moments are given by

$$\mathbf{F}_d(t) = \iint_S \Pi_j(\bar{x},\bar{y},\bar{z};t)\, \mathbf{n}\, dS, \quad \mathbf{G}_d(t) = \iint_S \Pi_j(\bar{x},\bar{y},\bar{z};t)\, (\mathbf{r}\times\mathbf{n})\, dS,$$

and S represents the surface given by

$$\bar{y} = \bar{c}(1 - \bar{x}^2/\bar{a}^2 - \bar{z}^2/\bar{b}^2)^{\frac{1}{2}}, \quad \bar{c} < 0.$$

By (2.1) and (2.10), $\mathbf{F}_d$ and $\mathbf{G}_d$ can be rewritten in terms of the dimensionless space variables and amplitudes as

$$\left.\begin{aligned} \mathbf{F}_d(t) &= \rho g \bar{a}^3\, \mathrm{Re}\left[X_j^0 \iint_S a u_j(x,y,z)\, \mathbf{n}\, dS\, e^{-i\sigma t}\right] \quad (j = 1,2,3), \\ \text{and} \quad \mathbf{G}_d(t) &= \rho g \bar{a}^4\, \mathrm{Re}\left[\theta_j^0 \iint_S a u_j(x,y,z)\, (\mathbf{r}\times\mathbf{n})\, dS\, e^{-i\sigma t}\right] \quad (j = 4,5,6). \end{aligned}\right\} \tag{4.2}$$

Now, expressing the dynamic forces and moments with a component in phase with the acceleration and that in phase with the velocity of the ellipsoid, we obtain

$$\left.\begin{aligned} \mathbf{F}_d(t) &= -\bar{\mathbf{M}}\ddot{\bar{X}}_j(t) - \bar{\mathbf{N}}\dot{\bar{X}}_j(t) \quad (j = 1,2,3), \\ \text{and} \quad \mathbf{G}_d(t) &= -\bar{\mathbf{I}}\ddot{\theta}_j(t) - \bar{\mathbf{H}}\dot{\theta}_j(t) \quad (j = 4,5,6). \end{aligned}\right\} \tag{4.3}$$

Then, the comparison of (4.2) with (4.3) yields

$$\left.\begin{aligned} \mathbf{M} &= \frac{\bar{\mathbf{M}}}{\rho \bar{a}^3} = \mathrm{Re}\left[\iint_S u_j(x,y,z)\, \mathbf{n}\, dS\right], \\ \mathbf{N} &= \frac{\bar{\mathbf{N}}}{\rho\sigma \bar{a}^3} = \mathrm{Im}\left[\iint_S u_j(x,y,z)\, \mathbf{n}\, dS\right] \quad (j = 1,2,3), \\ \mathbf{I} &= \frac{\bar{\mathbf{I}}}{\rho a^4} = \mathrm{Re}\left[\iint_S u_j(x,y,z)\, (\mathbf{r}\times\mathbf{n})\, dS\right], \\ \mathbf{H} &= \frac{\bar{\mathbf{H}}}{\rho\sigma \bar{a}^4} = \mathrm{Im}\left[\iint_S u_j(x,y,z)\, (\mathbf{r}\times\mathbf{n})\, dS\right] \quad (j = 4,5,6), \end{aligned}\right\} \tag{4.4}$$

where $\mathbf{M}$ and $\mathbf{N}$ denote the dimensionless added mass and linear damping coefficient, and $\mathbf{I}$ and $\mathbf{H}$ denote the dimensionless added moment of inertia and angular damping coefficient, respectively. In passing, we note that the quantities $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ in (4.3) can be related to the added mass coefficient $\bar{\mathbf{k}}$ and damping coefficient $2\bar{\mathbf{h}}$, which are employed by Ursell (1949*a*) and Havelock (1955) as

$$\bar{\mathbf{M}} = m\bar{\mathbf{k}}, \quad \text{and} \quad \bar{\mathbf{N}} = m\sigma(2\bar{\mathbf{h}}), \tag{4.5}$$

where m represents the actual mass of fluid displaced by a rigid body.

If a rigid body is a cylinder, the force $\mathbf{F}^{(2)}$ and moments $G^{(2)}$ arising from small forced oscillations are given by

$$\left.\begin{aligned} \mathbf{F}^{(2)}(t) &= \mathbf{F}_d^{(2)} - \hat{\mathbf{j}}\rho g \bar{X}_2(t)\, 2\bar{a} \quad (j = 1,2), \\ \text{and} \quad G^{(2)}(t) &= G_d^{(2)} - \rho g \theta_3(t)\, \tfrac{2}{3}\bar{a}(\bar{a}^2+\bar{b}^2), \end{aligned}\right\} \tag{4.6}$$

where the dynamic forces and moments are given by

$$\mathbf{F}_d^{(2)}(t) = \int_C \Pi_j^{(2)}(\bar{x},\bar{y};t)\,(\hat{\imath}n_x+\hat{\jmath}n_y)\,dC, \quad \text{and} \quad G_d^{(2)}(t) = \int_C \Pi_3^{(2)}(\bar{x},\bar{y};t)\,(\bar{x}n_y-\bar{y}n_x)\,dC,$$

and C represents the periphery of the cylinder, $\bar{y} = \bar{b}(1-\bar{x}^2/\bar{a}^2)^{\frac{1}{2}}$, $\bar{b} < 0$, and $\Pi_j^{(2)}, j = 1,2,3$, is the two-dimensional dynamic pressure. Rewriting the dynamic forces and moments in the form

$$\left.\begin{aligned} \mathbf{F}_d^{(2)} &= \rho g \bar{a}^2 \operatorname{Re}\left[X_j^0 \int_C a u_j(x,y)\,(\hat{\imath}n_x+\hat{\jmath}n_y)\,dC\,e^{-i\sigma t}\right] \quad (j=1,2),\\ \text{and} \qquad G_d^{(2)} &= \rho g \bar{a}^3 \operatorname{Re}\left[\theta_3^0 \int_C a u_3(x,y)\,(xn_y-yn_x)\,dC\,e^{-i\sigma t}\right], \end{aligned}\right\} \tag{4.7}$$

and identifying the component either in phase with the acceleration or the velocity from

$$\left.\begin{aligned} \mathbf{F}_d^{(2)}(t) &= -\overline{\mathbf{M}}^{(2)}\ddot{X}_j(t)-\overline{\mathbf{N}}^{(2)}\dot{X}_j(t) \quad (j=1,2),\\ \text{and} \qquad G_d^{(2)}(t) &= -\bar{I}_z^{(2)}\ddot{\theta}_3(t)-\bar{H}_z^{(2)}\dot{\theta}_3(t), \end{aligned}\right\} \tag{4.8}$$

we obtain the dimensionless added mass, added moment of inertia and damping coefficients for a two-dimensional problem as

$$\left.\begin{aligned} \mathbf{M}^{(2)} &= \frac{\overline{\mathbf{M}}^{(2)}}{\rho\bar{a}^2} = \operatorname{Re}\left[\int_C u_j(x,y)\,(\hat{\imath}n_x+\hat{\jmath}n_y)\,dC\right],\\ \mathbf{N}^{(2)} &= \frac{\overline{\mathbf{N}}^{(2)}}{\rho\sigma\bar{a}^2} = \operatorname{Im}\left[\int_C u_j(x,y)\,(\hat{\imath}n_x+\hat{\jmath}n_y)\,dC\right] \quad (j=1,2),\\ I_z^{(2)} &= \frac{\bar{I}_z^{(2)}}{\rho\bar{a}^3} = \operatorname{Re}\left[\int_C u_3(x,y)\,(xn_y-yn_x)\,dC\right],\\ H_z^{(2)} &= \frac{\bar{H}_z^{(2)}}{\rho\sigma\bar{a}^3} = \operatorname{Im}\left[\int_C u_3(x,y)\,(xn_y-yn_x)\,dC\right]. \end{aligned}\right\} \tag{4.9}$$

It is often necessary to estimate the three-dimensional physical quantities knowing only the two-dimensional results. For instance, using the hydrodynamic quantities of the cylinder (4.9), an attempt can be made to evaluate the same quantities of the ellipsoid (4.4). A method of relating the two-dimensional results to the three-dimensional estimates is called the 'strip method'. In the present problem, the added mass, M_y^s, added moment of inertia, I_x^s, and damping coefficients, N_y^s and H_x^s, of an ellipsoid for modes of heave and roll can be evaluated by the following strip method formulae (for the derivation, see Kim 1963, Appendix 2):

$$\left.\begin{aligned} M_y^s &= 2\int_0^1 b^2(1-x^2)\,M_y^{(2)}[ab(1-x^2)^{\frac{1}{2}}]\,dx,\\ N_y^s &= 2\int_0^1 b^2(1-x^2)\,N_y^{(2)}[ab(1-x^2)^{\frac{1}{2}}]\,dx,\\ I_x^s &= 2\int_0^1 b^4(1-x^2)^2\,I_x^{(2)}[ab(1-x^2)^{\frac{1}{2}}]\,dx,\\ H_x^s &= 2\int_0^1 b^4(1-x^2)^2\,H_x^{(2)}[ab(1-x^2)^{\frac{1}{2}}]\,dx, \end{aligned}\right\} \tag{4.10}$$

where $M_y^{(2)}$, $N_y^{(2)}$, $I_x^{(2)}$ and $H_x^{(2)}$ denote the data of the cylinder. It will be shown presently how the estimates obtained by (4.10) compare with the corresponding three-dimensional data.

5. Numerical procedure

We are ultimately concerned with the solution of equations (3.7) and (3.8) for arbitrary values of a. Here (3.7) deals with an ellipsoid $\bar{x}^2/\bar{a}^2+\bar{y}^2/\bar{c}^2+\bar{z}^2/\bar{b}^2 = 1$, $\bar{c} < 0$, which has length, beam and draft of $2\bar{a}$, $2\bar{b}$ and $|\bar{c}|$ while (3.8) deals with a cylinder having a cross-section $\bar{x}^2/\bar{a}^2+\bar{y}^2/\bar{b}^2 = 1$, $\bar{b} < 0$, in which $2\bar{a}$ and $|\bar{b}|$ denote beam and draft. Suitable co-ordinates which conform to these body forms are the ellipsoidal polar co-ordinates

$$\left.\begin{aligned} x &= \cos\alpha\sin\phi, \quad y = c\cos\phi, \quad z = b\sin\alpha\sin\phi, \\ \xi &= \cos\beta\sin\psi, \quad \eta = c\cos\psi, \quad \zeta = b\sin\beta\sin\psi, \end{aligned}\right\} \tag{5.1}$$

and the cylindrical polar co-ordinates

$$\left.\begin{aligned} x &= \cos\theta, \quad y = b\sin\theta, \\ \xi &= \cos\epsilon, \quad \eta = b\sin\epsilon, \end{aligned}\right\} \tag{5.2}$$

where x, y, z and ξ, η, ζ are dimensionless space variables, and b, c are dimensionless lengths resulting from dividing each physical length by a typical length of the body $\bar{a}$.

By use of the new co-ordinates we can express equations (3.7) and (3.8) in the form

$$-F_j(\alpha,\phi)+\frac{bc}{2\pi}\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} F_j(\beta,\psi)\frac{\partial}{\partial n}G(\alpha,\phi;\beta,\psi)\,T(\beta,\psi)\sin\psi\,d\psi\,d\beta = 2H_j(\alpha,\phi), \tag{5.3}$$

and

$$-F_j(\theta)+\frac{1}{\pi}\int_0^{\pi} F_j(\epsilon)\frac{\partial}{\partial n}G(\theta;\epsilon)\,R(\epsilon)d\epsilon = 2H_j(\theta), \tag{5.4}$$

where

$$\left.\begin{aligned} T(\alpha,\phi) &= [(\cos^2\alpha+\sin^2\alpha/b^2)\sin^2\phi+\cos^2\phi/c^2]^{\frac{1}{2}}, \\ \text{and}\quad R(\theta) &= (\sin^2\theta+b^2\cos^2\theta)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.5}$$

We restrict ourselves to the three-dimensional case since the two-dimensional equation is solved in an analogous manner. The method is based on Fredholm's procedure of replacing the integral equation with a finite set of linear equations. Suppose the β- and ψ-axes which bound the region of integration $S(\beta,\psi)$ are divided, respectively, into 24 and 6 equal divisions. Then, a lattice can be formed on $S(\beta,\psi)$ by connecting the points of divisions with straight lines parallel to the β- and ψ-axes. The element of such a lattice is a square having the side $h = \frac{1}{12}\pi$. Now, choosing the centroids of the elements as pivotal points, equation (5.3) is to be solved on these discrete points. Since F_j, the source strength, is a complex function, equation (5.3) can be separated into a pair of equations for the real and imaginary parts:

$$\left.\begin{aligned} &-\operatorname{Re}[F_j(\alpha,\phi)]+\frac{bc}{2\pi}\sum_{l=1}^{24}\sum_{m=1}^{6}\left\{\operatorname{Re}[F_j(\beta_l\psi_m)]\operatorname{Re}\left[\frac{\partial G}{\partial n}\right]-\operatorname{Im}[F_j(\beta_l\psi_m)]\operatorname{Im}\left[\frac{\partial G}{\partial n}\right]\right\} \\ &\qquad\qquad\times T(\beta_l\psi_m)\sin\psi_m = 2H_j(\alpha,\phi), \\ &-\operatorname{Im}[F_j(\alpha,\phi)]+\frac{bc}{2\pi}\sum_{l=1}^{24}\sum_{m=1}^{6}\left\{\operatorname{Re}[F_j(\beta_l\psi_m)]\operatorname{Im}\left[\frac{\partial G}{\partial n}\right]+\operatorname{Im}[F_j(\beta_l\psi_m)]\operatorname{Re}\left[\frac{\partial G}{\partial n}\right]\right\} \\ &\qquad\qquad\times T(\beta_l\psi_m)\sin\psi_m = 0, \end{aligned}\right\} \tag{5.6}$$

where, as has been noted earlier, the integral involving $(\partial/\partial n)R^{-1}$, R being the distance between a point source and a variable point, in Re $[\partial G/\partial n]$ is an improper integral. Nevertheless, this integral can be shown to exist.

We turn to the evaluation of the following integral

$$I_1 = bc\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} F(\beta,\psi)\,\frac{\partial}{\partial n}\frac{1}{R}\,T(\beta,\psi)\sin\psi\,d\psi\,d\beta, \tag{5.7}$$

where
$$\frac{\partial}{\partial n}\frac{1}{R} = \mathbf{n}(\alpha,\phi)\,\frac{\Xi(\beta,\psi)-\mathbf{X}(\alpha,\phi)}{|\Xi(\beta,\psi)-\mathbf{X}(\alpha,\phi)|^3},$$

with
$$\mathbf{X}(\alpha,\phi) = \hat{\mathbf{i}}x(\alpha,\phi)+\hat{\mathbf{j}}y(\alpha,\phi)+\hat{\mathbf{k}}z(\alpha,\phi),$$
$$\Xi(\beta,\psi) = \hat{\mathbf{i}}\xi(\beta,\psi)+\hat{\mathbf{j}}\eta(\beta,\psi)+\hat{\mathbf{k}}\zeta(\beta,\psi),$$

and $\mathbf{n}(\alpha,\phi)$ being a unit normal on the surface $S(\beta,\psi)$. Let us suppose $\mathbf{X}(\alpha,\phi)$ and $\Xi(\beta,\psi)$ represent the position vectors of a point source and a variable point in its proximity. Then, by Taylor's expansion, we may write

$$\Xi(\beta,\psi) = \mathbf{X}(\alpha,\phi)+\sum_{n=1}^{\infty}\frac{1}{n!}(\Delta\alpha D_\alpha+\Delta\phi D_\phi)^n\,\mathbf{X}(\alpha,\phi), \tag{5.8}$$

with
$$\Delta\alpha = \delta\cos\tau,\quad \Delta\phi = \delta\sin\tau,\quad\text{and}\quad \delta = [(\beta-\alpha)^2+(\phi-\psi)^2]^{\frac{1}{2}}, \tag{5.9}$$

where $\tau = \tan^{-1}(\psi-\phi)/(\beta-\alpha)$. By the use of (5.8) and the binomial expansion, it can be shown that (Kim 1964, Appendix)

$$\frac{\partial}{\partial n}\frac{1}{R} \sim \frac{1}{T(\alpha,\phi)}\left[\frac{1}{\delta}A_1+B_1+O(\delta)\right], \tag{5.10}$$

where
$$A_1 = -\frac{(\Delta^2\alpha+\Delta^2\phi)^{\frac{1}{2}}}{2}\,\frac{\Delta^2\alpha\sin^2\phi+\Delta^2\phi}{(E+c^2\Delta^2\phi\sin^2\phi)^{\frac{3}{2}}},$$

$$B_1 = \frac{3\Omega(\Delta^2\alpha\sin^2\phi+\Delta^2\phi)-\Delta^2\alpha\Delta\phi\sin 2\phi}{4(E+c^2\Delta^2\phi\sin^2\phi)^{\frac{3}{2}}},$$

with
$$\Omega = \frac{(\Delta\alpha\mathbf{X}_\alpha+\Delta\phi\mathbf{X}_\phi)\,(\Delta^2\alpha\mathbf{X}_{\alpha\alpha}+2\Delta\alpha\Delta\phi\mathbf{X}_{\alpha\phi}+\Delta^2\phi\mathbf{X}_{\phi\phi})}{\Delta^2\alpha\mathbf{X}_\alpha^2+2\Delta\alpha\Delta\phi\mathbf{X}_\alpha.\mathbf{X}_\phi+\Delta^2\phi\mathbf{X}_\phi^2},$$

and $E = E(\Delta\alpha,\Delta\phi)$

$$= (\Delta\alpha\sin\alpha\sin\phi-\Delta\phi\cos\alpha\cos\phi)^2+b^2(\Delta\alpha\cos\alpha\sin\phi+\Delta\phi\sin\alpha\sin\phi)^2.$$

It follows then that

$$\lim_{R\to 0}\left[\frac{\partial}{\partial n}\frac{1}{R}-\frac{1}{T(\alpha,\phi)}\left(\frac{1}{\delta}A_1+B_1\right)\right] = 0. \tag{5.11}$$

Thus, on the transformed plane as $\Xi(\beta,\psi)$ tends to $\mathbf{X}(\alpha,\phi)$, the function $(\partial/\partial n)R^{-1}$ possesses a part which involves the reciprocal distance, and an indeterminate part which depends upon the angle of approach.

By (5.7) and (5.9), the part of the integral I_1, taken over the neighbourhood element Δ, can be approximated as

$$[I_1]_\Delta \approx bc\sin\phi F(\alpha,\phi)\int_0^{2\pi}\int_0^{h(\tau)}\left(\frac{A_1}{\delta}+B_1\right)d\delta\,\delta\,d\tau$$
$$+bcF(\alpha,\phi)\int_{\alpha-\frac{1}{2}h}^{\alpha+\frac{1}{2}h}\int_{\phi-\frac{1}{2}h}^{\phi+\frac{1}{2}h}\left[\frac{\partial}{\partial n}\frac{1}{R}-\frac{1}{T(\alpha,\phi)}\left(\frac{A_1}{\delta}+B_1\right)\right]T(\beta,\psi)\sin\psi\,d\psi\,d\beta$$
$$\approx -\frac{h^2}{4}bcF(\alpha,\phi)\left\{\frac{2}{h}\sin\phi\left[\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}\frac{|\sec\tau|\,(\cos^2\tau\sin^2\phi+\sin^2\tau)\,d\tau}{(E+c^2\sin^2\tau\sin^2\phi)^{\frac{3}{2}}}\right.\right.$$
$$\left.+\int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}\frac{|\csc\tau|\,(\cos^2\tau\sin^2\phi+\sin^2\tau)\,d\tau}{(E+c^2\sin^2\tau\sin^2\phi)^{\frac{3}{2}}}\right]$$
$$+\sum_{i=1}^{3}\sum_{j=1}^{3}C_iD_j\left[\frac{1}{R_{ij}^3}[1-\cos(\alpha-\beta_i)\sin\phi\sin\psi_j-\cos\phi\cos\psi_j]\right.$$
$$\left.\left.-\frac{(\beta_i-\alpha)^2\sin^2\phi+(\psi_j-\phi)^2}{[E_{ij}+c^2(\psi_j-\phi)^2\sin^2\phi]^{\frac{3}{2}}}\right]\frac{T(\beta_i\psi_j)}{T(\alpha,\phi)}\sin\psi_j\right\}, \tag{5.12}$$

where the δ-integration is carried out explicitly with $h(\tau)$ given by

$$h(\tau)=\begin{matrix}\frac{1}{2}h\,|\sec\tau|\\ \frac{1}{2}h\,|\csc\tau|\end{matrix}\quad\text{for}\quad\left.\begin{matrix}-\frac{1}{4}\pi;\ \frac{3}{4}\pi\leqslant\tau\leqslant\frac{1}{4}\pi;\ \frac{5}{4}\pi,\\ \frac{1}{4}\pi;\ \frac{5}{4}\pi\leqslant\tau\leqslant\frac{3}{4}\pi;\ \frac{7}{4}\pi,\end{matrix}\right\}$$

and the second integral is evaluated using Simpson's rule since its integrand vanishes at $(\beta,\psi)=(\alpha,\phi)$. Furthermore, it should be noted that

$$\int_0^{2\pi}\int_0^{h(\tau)}B_1\,d\delta\,\delta\,d\tau=0,\quad\text{since}\quad B_1(\pi+\tau)=-B_1(\tau).$$

The remaining part of the integral I_1 taken over $S-\Delta$ can be approximated by use of Simpson's rule.

Application of the formula (5.12) and Simpson's rule enables us to write out the double summations in (5.6) as a linear combination of $\mathrm{Re}\,[F_j]$ and $\mathrm{Im}\,[F_j]$ so that we can obtain a finite set of linear equations for each mode of oscillation.

By (5.1), the right-hand members for the six degrees of freedom are obtained from (2.12) as

$$\left.\begin{aligned}&H_1(\alpha,\phi)=\frac{\cos\alpha\sin\phi}{T(\alpha,\phi)},\quad H_2=\frac{\cos\phi}{cT(\alpha,\phi)},\quad H_3(\alpha,\phi)=\frac{\sin\alpha\sin\phi}{bT(\alpha,\phi)},\\&H_4(\alpha,\phi)=\frac{c^2-b^2}{bc}\frac{\sin\alpha\sin\phi\cos\phi}{T(\alpha,\phi)},\quad H_5(\alpha,\phi)=\frac{b^2-1}{b}\frac{\sin\alpha\cos\alpha\sin^2\phi}{T(\alpha,\phi)}\\&\text{and}\quad H_6(\alpha,\phi)=\frac{1-c^2}{c}\frac{\cos\alpha\sin\phi\cos\phi}{T(\alpha,\phi)}.\end{aligned}\right\} \tag{5.13}$$

Here the strength of source F_j in the problem depends not only upon the mode and frequency of the oscillation, but also upon the form of the body. Since the immersed surface of an ellipsoid possesses symmetry about the $\bar{x}$ and $\bar{z}$ axes, we expect F_j to exhibit the corresponding symmetry. However, due to the transformation of co-ordinates by (5.1) this symmetry is to be expressed in terms of the variable α. For the specific modes of oscillation, we note that

$$\left.\begin{aligned} F_{1,6}(\alpha,\phi) &= F_{1,6}(-\alpha,\phi) = -F_{1,6}(\pi-\alpha,\phi) = -F_{1,6}(\pi+\alpha,\phi), \\ F_2(\alpha,\phi) &= F_2(-\alpha,\phi) = F_2(\pi-\alpha,\phi) = F_2(\pi+a,\phi), \\ F_{3,4}(\alpha,\phi) &= -F_{3,4}(-\alpha,\phi) = F_{3,4}(\pi-\alpha,\phi) = -F_{3,4}(\pi+\alpha,\phi), \\ F_5(\alpha,\phi) &= -F_5(-\alpha,\phi) = -F_5(\pi-\alpha,\phi) = F_5(\pi+\alpha,\phi). \end{aligned}\right\} \tag{5.14}$$

On the basis of such symmetric properties, it is sufficient to consider the linear equations (5.6) only at 36 pivotal points contained in one quadrant. Furthermore, relating the values of F_j in other quadrants by (5.14), a rectangular matrix produced by the double summation of (5.6) should be folded into a square-form matrix for the solution of $\mathrm{Re}\,[F_j]$ and $\mathrm{Im}\,[F_j]$ from 72 linear equations.

Having found the real and imaginary parts of the source strength F_j, the pressure function u_j can be determined from

$$\begin{aligned} u_j(\alpha,\phi) \approx \frac{bc}{4\pi}\sum_{l=1}^{24}\sum_{m=1}^{6} [\{\mathrm{Re}\,[F_j(\beta_l\psi_m)]\,\mathrm{Re}\,[G] - \mathrm{Im}\,[F_j(\beta_l\psi_m)]\,\mathrm{Im}\,[G]\} \\ + i\{\mathrm{Re}\,[F_j(\beta_l\psi_m)]\,\mathrm{Im}\,[G] + \mathrm{Im}\,[F_j(\beta_l\psi_m)]\,\mathrm{Re}\,[G]\}]\, T(\beta_l\psi_m)\sin\psi_m, \end{aligned} \tag{5.15}$$

where an approximation formula for the improper integral

$$I_2 = bc\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} F(\beta,\psi)\,\frac{1}{R}\,T(\beta,\psi)\sin\psi\,\mathrm{d}\psi\,d\beta \tag{5.16}$$

in $\mathrm{Re}\,[G]$ may be derived repeating the same reasoning employed in connexion with the improper integral I_1. Here we find that

$$\lim_{R\to 0}\left[\frac{1}{R} - \left(\frac{1}{\delta}A_2 + B_2\right)\right] = 0, \tag{5.17}$$

where $$A_2 = \frac{(\Delta^2\alpha+\Delta^2\phi)^{\frac{1}{2}}}{(E+c^2\Delta^2\phi\sin^2\phi)^{\frac{1}{2}}},\quad B_2 = -\frac{1}{2}\frac{\Omega}{(E+c^2\Delta^2\phi\sin^2\phi)^{\frac{1}{2}}}.$$

It follows then that

$$\begin{aligned} [I_2]_\Delta \approx \tfrac{1}{4}h^2 bcF(\alpha,\phi)\Bigg\{\frac{4}{h}\sin\phi\, T(\alpha,\phi)\Bigg[\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}\frac{|\sec\tau|\,d\tau}{(E+c^2\sin^2\tau\sin^2\phi)^{\frac{1}{2}}} \\ + \int_{\frac{1}{4}\pi}^{\frac{3}{4}\pi}\frac{|\csc\tau|\,d\tau}{(E+c^2\sin^2\tau\sin^2\phi)^{\frac{1}{2}}}\Bigg] \\ + \sum_{i=1}^{3}\sum_{j=1}^{3} C_i D_j\left[\frac{1}{R_{ij}} - \frac{1}{[E_{ij}+c^2(\psi_j-\phi)^2\sin^2\phi]}\right] T(\beta_i\psi_j)\sin\psi_j\Bigg\}, \end{aligned} \tag{5.18}$$

where $$\int_0^{2\pi}\int_0^{h(\tau)} B_2\,d\delta\,\delta\,d\tau = 0, \quad \text{since} \quad B_2(\pi+\tau) = -B_2(\tau).$$

Again, the remaining part of the integral I_2 taken over $S-\Delta$ can be approximated by use of Simpson's rule. We remark here that the use of the symmetry (5.14) facilitates the evaluation of the pressure function u_j by (5.15) for a given mode of oscillation.

6. Results and discussion

In this paper, the three-dimensional problem was solved for a half-ellipsoid having beam to length ratio $\bar{b}/\bar{a} = \frac{1}{4}$; and its special case, a half-spheroid $\bar{b}/\bar{a} = 1$. Consideration was given to the effect of draft by varying the half-length to draft ratio $\bar{a}/\bar{c}$ as $H = 4, 2, 1$ and $\frac{1}{2}$ for the spheroid, $H = 4$ and 2 for the $\frac{1}{4}$ ellipsoid. Furthermore, the two-dimensional problem was solved for a half-cylinder of an elliptic cross-section having half-beam to draft ratios $\bar{a}/\bar{b}$, $H = 4$, 2, 1 and $\frac{1}{2}$. The computation was generally performed with the values of the frequency

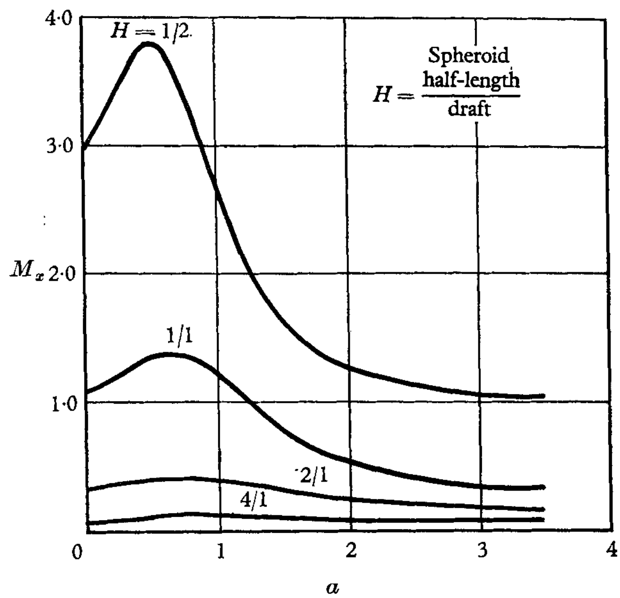

FIGURE 1. Added mass for surging (or swaying) spheroids $M_x = \bar{M}_x/\rho\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

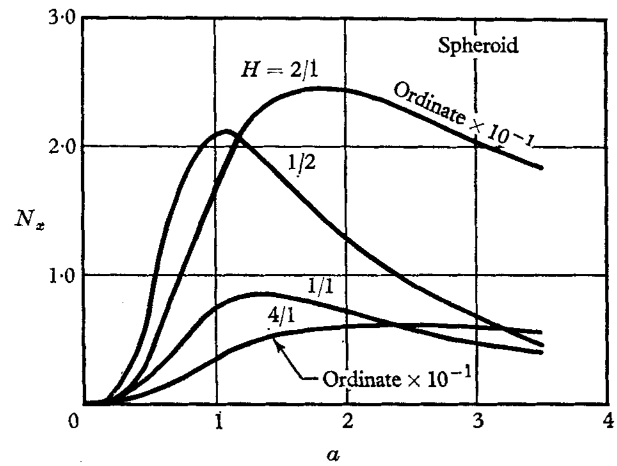

FIGURE 2. Damping coefficient for surging (or swaying) spheroids $N_x = \bar{N}_x/\rho\sigma\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

parameter $a = 0, 0{\cdot}10, 0{\cdot}25, 0{\cdot}50, 0{\cdot}75, 1{\cdot}0, 1{\cdot}5, 2{\cdot}0, 2{\cdot}5, 3{\cdot}0$ and $3{\cdot}5$ (except when more points were necessary due to a rapid change of curvature). According to the definition of the frequency parameter the minimum and maximum values of a correspond, respectively, to the cases in which the length of the wave generated

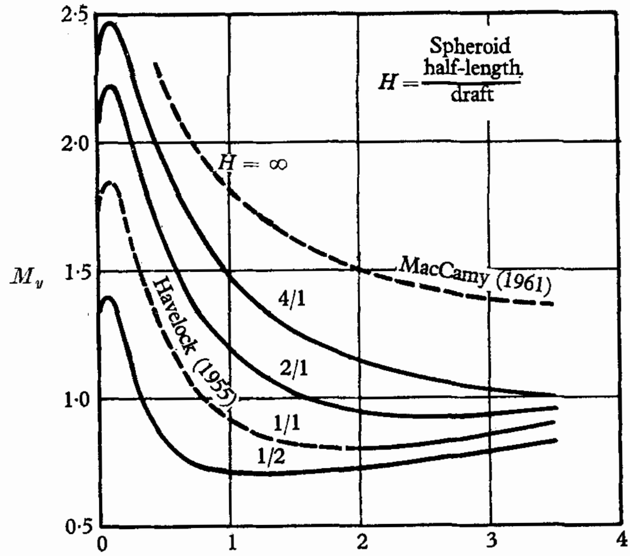

FIGURE 3. Added mass for heaving spheroids $M_y = \overline{M}_y/\rho\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

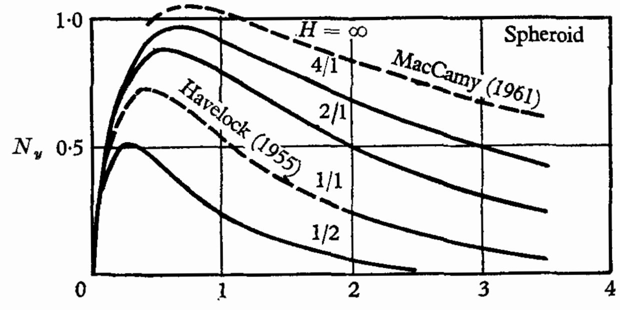

FIGURE 4. Damping coefficient for heaving spheroids $N_y = \overline{N}_y/\rho\sigma\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

by forced oscillation is approximately equal to 30 and to 1 times the length of the ellipsoid (or the beam of the cylinder). From the asymptotic behaviour of the kernel of the integral equation, it can be seen that as the value of a becomes large the oscillation of the kernel grows rapidly. For this reason, if the value of a is increased beyond the present range, a quite large number of pivotal points are required. Therefore, higher frequency results could be obtained with more computational labour, but it would be profitable to devise a different technique in line with Ursell's approach (1953). He obtained the higher frequency asymptotics from the solution of an integral equation in which the boundary values of a wave-potential occurs as unknown.

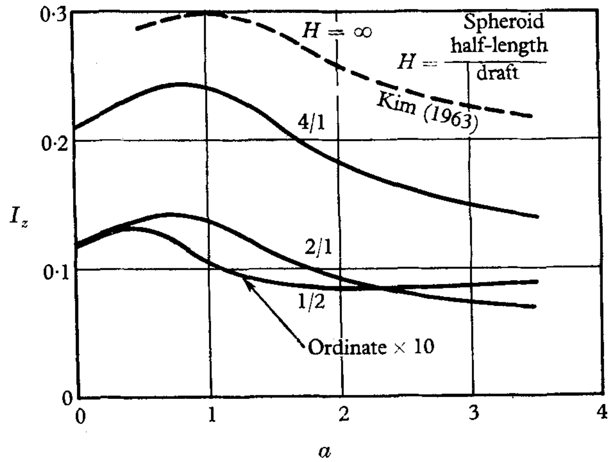

FIGURE 5. Added moment of inertia for pitching (or rolling) spheroids $I_z = \bar{I}_z/\rho\bar{a}^4$ as a function of $a = \bar{a}\sigma^2/g$.

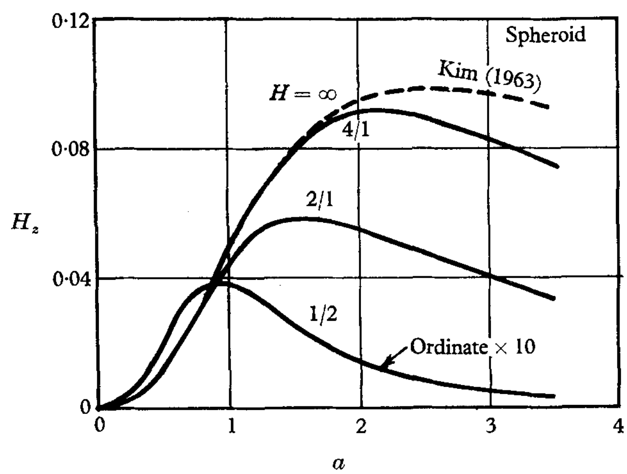

FIGURE 6. Damping coefficient for pitching (or rolling) spheroids $H_z = \bar{H}_z/\rho\sigma\bar{a}^4$ as a function of $a = \bar{a}\sigma^2/g$.

For each combination of the parameters $\bar{b}/\bar{a}$, H and a of an ellipsoid (or H and a of a cylinder) several sets of linear equations describing specific modes of oscillation were solved. Then, from the solutions of the linear equations, the pivotal values of the pressure were determined. Subsequently, summation of the real and imaginary parts of these pressures over the immersed surface by Simpson's rule yielded the hydrodynamic quantities. In figures 1–6, the dependence of the hydrodynamic quantities on the frequency parameter a for the spheroids having various drafts are presented. M_x and N_x represent the normalized added mass and associated damping coefficient for surge, M_y and N_y represent the normalized added mass and associated damping coefficient for heave, and I_z and H_z represent the normalized added moment of inertia and associated angular damping coefficient for pitch, respectively.

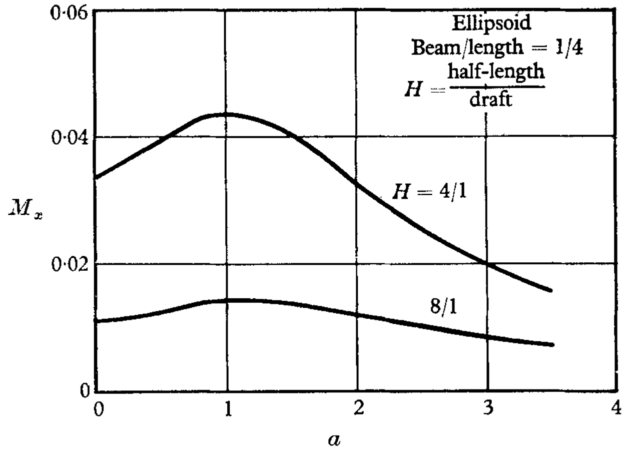

FIGURE 7. Added mass for surging ellipsoids $M_x = \overline{M}_x/\rho\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

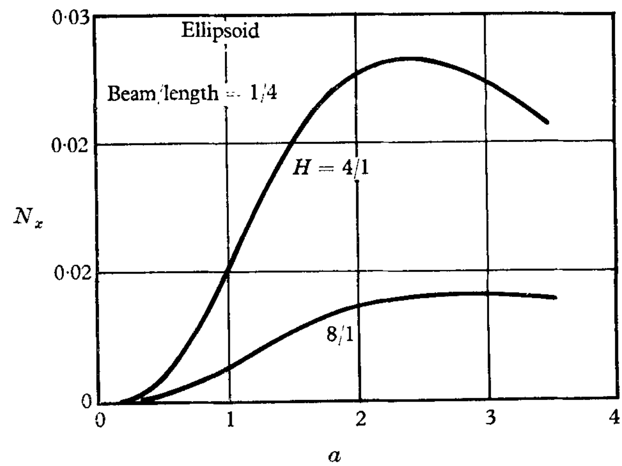

FIGURE 8. Damping coefficients for surging ellipsoids $N_x = \overline{N}_x/\rho\sigma\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

The dependence on the parameter a of the same quantities for the $\frac{1}{4}$ ellipsoid having various drafts is presented in figures 7–12. The last set of figures (figures 13–18) elucidate how two-dimensional hydrodynamic quantities vary with the parameter a and with the draft. Here $M_x^{(2)}$ and $N_x^{(2)}$ represent the normalized added mass and associated damping coefficient for sway, and $I_z^{(2)}$ and $H_z^{(2)}$ represent the normalized added moment of inertia and associated angular damping coefficient for roll.

In figures 3 and 4, and figures 15 and 16, variation of M_y and N_y for spheroids and that of $M_y^{(2)}$ and $N_y^{(2)}$ for cylinders are shown. The problem of a heaving cylinder was first worked out by Ursell (1949*a*). He made use of a wave potential which consists of a set of non-orthogonal polynomials and a suitable point source

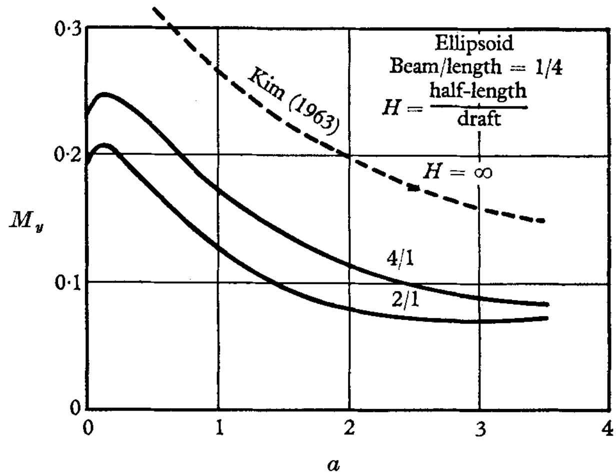

FIGURE 9. Added mass for heaving ellipsoids $M_y = \overline{M}_y/\rho\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

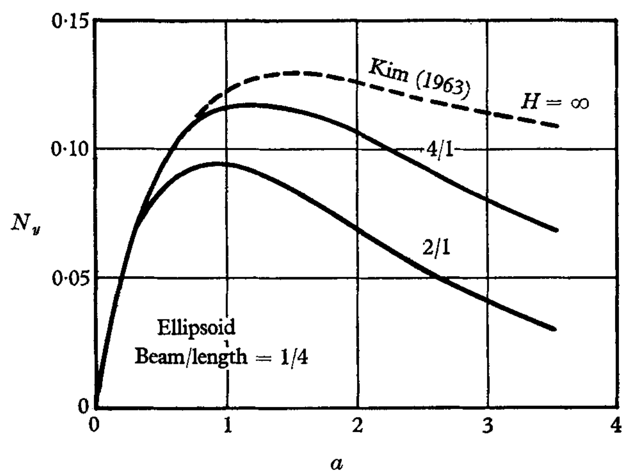

FIGURE 10. Damping coefficient for heaving ellipsoids $N_y = \overline{N}_y/\rho\sigma\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

at the origin on the physical ground that the forced oscillation of a rigid body produces a standing wave in its vicinity and a progressive wave at a large distance from the body. Ursell's method has been employed by Porter (1960) for the study of heaving cylinders having an elliptic cross-section. The extension of Ursell's method to the three-dimensional problem of a heaving spheroid can also be seen in papers of MacCamy (1954), Havelock (1955) and Barakat (1962).

The broken curves in the two sets of drawings, figures 3 and 4, and figures 15 and 16, indicate the present results, which indicate good agreement with the previous results obtained respectively by Havelock, Ursell and Porter as so

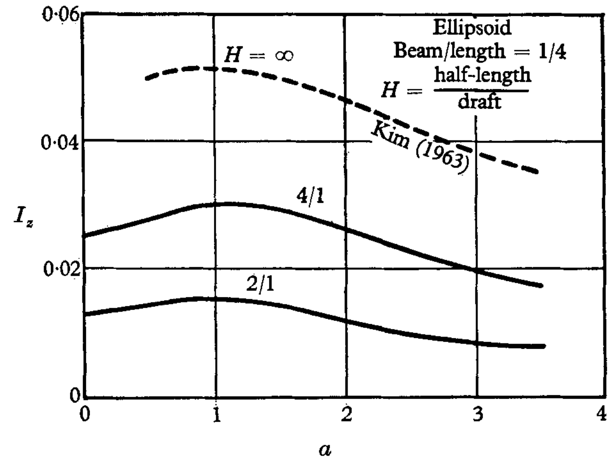

FIGURE 11. Added moment of inertia for pitching ellipsoids $I_z = \bar{I}_z/\rho\bar{a}^4$ as a function of $a = \bar{a}\sigma^2/g$.

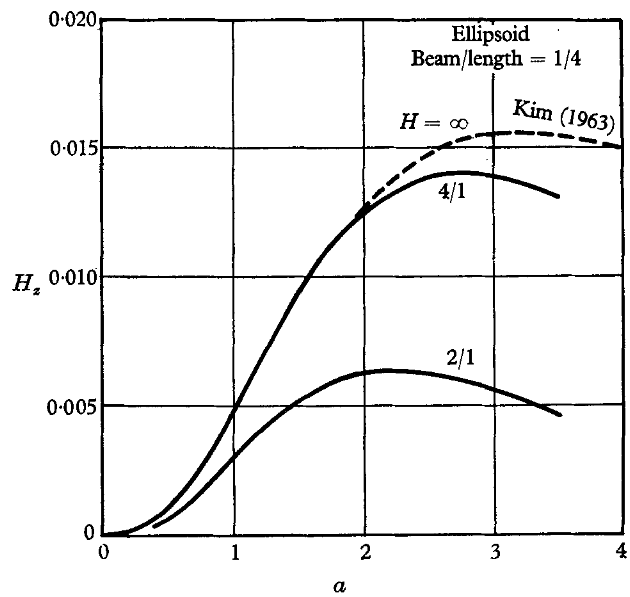

FIGURE 12. Damping coefficient for pitching ellipsoids $H_z = \bar{H}_z/\rho\sigma\bar{a}^4$ as a function of $a = \bar{a}\sigma^2/g$.

labelled. The curves attributed to MacCamy in both sets correspond to the first-order solutions of the shallow-draft approximation applied to a rigid body with small draft (see MacCamy 1961). In effect, those represent exact solutions of a heaving circular disk in figures 3 and 4, and of a heaving plate in figures 15 and

16. We note that M_y, N_y and $N_y^{(2)}$ increase systematically as the draft of the body decreases. In a limiting case $H = \infty$ when the draft is zero, the wave-making effect attains its maximum.

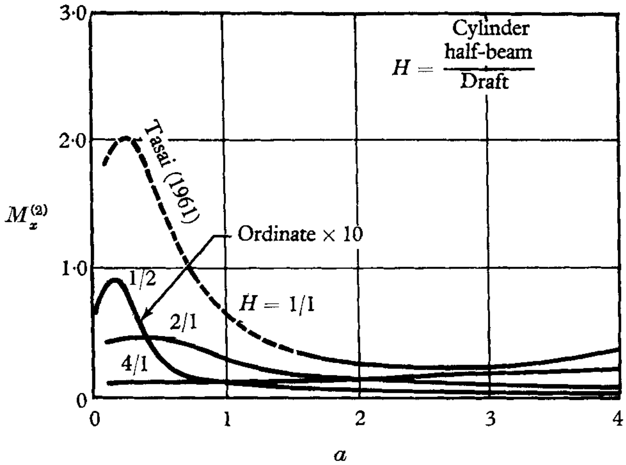

FIGURE 13. Added mass for swaying cylinders $M_x^{(2)} = \bar{M}_x^{(2)}/\rho\bar{a}^2$ as a function of $a = \bar{a}\sigma^2/g$.

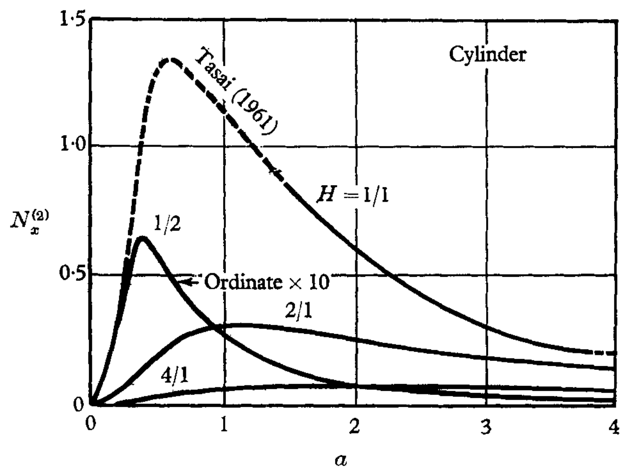

FIGURE 14. Damping coefficient for swaying cylinders $N_x^{(2)} = \bar{N}_x^{(2)}/\rho\sigma\bar{a}^2$ as a function of $a = \bar{a}\sigma^2/g$.

The low-frequency asymptotics of three-dimensional potentials can be shown (Kim 1964) as

$$u^{(3)}(0) = \lim_{a\to 0} u(x,y,z) = \frac{1}{4\pi}\iint_S f_{00}\left(\frac{1}{R}+\frac{1}{R'}\right) dS + i0 \tag{6.1}$$

where f_{00} is the solution of

$$-f_{00} + \frac{1}{2\pi}\iint_S f_{00}\frac{\partial}{\partial n}\left(\frac{1}{R}+\frac{1}{R'}\right) dS = 2h,$$

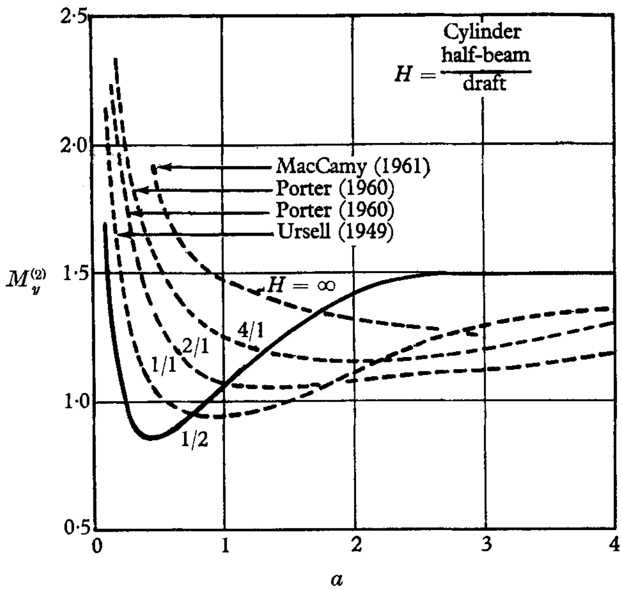

FIGURE 15. Added mass for heaving cylinders $M_y^{(2)} = \bar{M}_y^{(2)}/\rho\bar{a}^2$ as a function of $a = \bar{a}\sigma^2/g$.

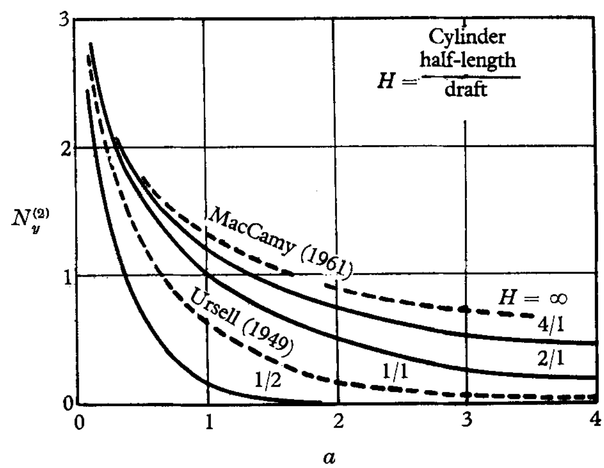

FIGURE 16. Damping coefficient for heaving cylinders $N_y^{(2)} = \bar{N}_y^{(2)}/\rho\sigma\bar{a}^2$ as a function of $a = \bar{a}\sigma^2/g$.

Hence, the limiting value of M_y which was evaluated using the real part of $u^{(3)}(0)$ becomes a non-zero constant, while the limiting value of N_y which uses the imaginary part of $u^{(3)}(0)$ vanishes. For a heaving sphere Ursell† found that the high-frequency asymptotics depend upon the potential

$$u^{(3)}(\infty) \sim -i3a^{-1}\,e^{-ia}\,r^{-\frac{1}{2}}\,e^{ay+iar} \quad \text{as} \quad r \to \infty, \tag{6.2}$$

† Prof. Ursell provided the author with his high-frequency results. Here, expressions (6.2) and (6.3) were obtained by converting Prof. Ursell's results according to the normalization used in the present paper.

and it follows that

$$\lim_{a\to\infty} M_y \sim \pi\left(\frac{1}{3}-\frac{1}{8a}\right), \quad \text{and} \quad \lim_{a\to\infty} N_y \sim \frac{9\pi}{a^4}. \tag{6.3}$$

We remark that as the present computation ranges up to the wavelength $\bar{\lambda}$ equal to the length of the sphere $2\bar{a}$, the high-frequency asymptotics (6.3) cannot be

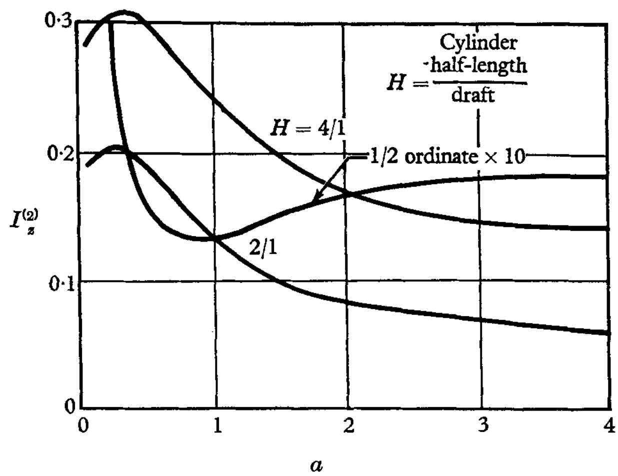

FIGURE 17. Added moment of inertia for rolling cylinders $I_z^{(2)} = \bar{I}_z^{(2)}/\rho\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

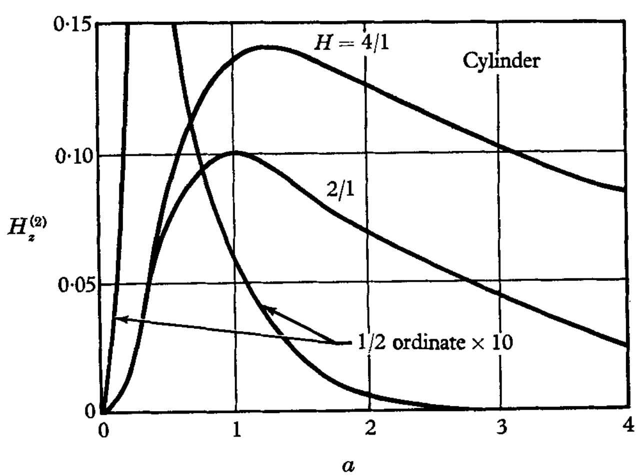

FIGURE 18. Damping coefficient for rolling cylinders $H_z^{(2)} = \bar{H}_z^{(2)}/\rho\sigma\bar{a}^3$ as a function of $a = \bar{a}\sigma^2/g$.

joined to the present results. However, we find that in the range of the parameter $a = 3.0 \sim 4.0$ the result given by (6.3) lies slightly above the present M_y ($H = 1$) curve.

In figures 15 and 16, it should be observed that due to the property of the potential $u^{(2)}(0)$, which also can be shown as

$$u^{(2)}(0) = \lim_{a\to 0} u(x,y) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_C f_{00}(\ln\varpi+\ln\varpi')\,dC+i0 \\ \quad \text{if } f_{00} \text{ is an odd function of } x, \\ O(\ln a) - i\displaystyle\int_C f_{00}\,dC \\ \quad \text{if } f_{00} \text{ is an even function of } x, \end{cases} \tag{6.4}$$

where f_{00} is the solution of the two-dimensional equation

$$-f_{00}+\frac{1}{\pi}\int_C f_{00}\frac{\partial}{\partial n}(\ln\varpi+\ln\varpi')\,dC = 2h,$$

the limiting value of $M_y^{(2)}$ becomes logarithmically infinite while that of $N_y^{(2)}$ becomes a non-zero constant. Havelock has attributed the infinite value of $M_y^{(2)}$ to the fact that when $a = 0$, the condition at the free surface, i.e. $u_y = 0$, makes the two-dimensional problem indeterminate. We emphasize that the different behaviour of the lower asymptotics in the two- and three-dimensional problems exhibited by (6.1) and (6.4) deprives the 'strip method' of being a useful way of estimating the three-dimensional hydrodynamic quantities from the two-dimensional data. For a circular cylinder Ursell (1953) obtained the high-frequency asymptotics as

$$u^{(2)}(\infty) \sim -i4a^{-2}\,e^{-ia}\,e^{ay+iax} \quad \text{as} \quad |x|\to\infty, \tag{6.5}$$

and it follows that

$$\lim_{a\to\infty} M_y^{(2)} \sim \left(\frac{\pi}{2}-\frac{2}{3a}\right), \quad \text{and} \quad \lim_{a\to\infty} N_y^{(2)} \sim \frac{16}{a^4}. \tag{6.6}$$

Note that the $M_y^{(2)}(H = 1)$ curve in figure 15 lies slightly below that given by (6.6).

Next, let us look at the two sets of drawings, figures 1 and 2, and figures 13 and 14, which present, for the sway mode, the normalized added mass and associated damping coefficient of spheroids and those of cylinders as a function of the parameter a. In the case of a spheroid, sway and surge are the same mode. By the use of Ursell's method, which can be used conveniently whenever the mapping of the cross-section of a cylinder on a circle is known explicitly, Tasai (1961) obtained the added mass coefficient $\bar{k}_x^{(2)} = [2H/\pi]\,M_x^{(2)}$ and the amplitude ratio $A_x^{(2)} = a[N_x^{(2)}]^{\frac{1}{2}}$ up to the value of $a = 1{\cdot}5$. The present results are in complete agreement with Tasai's results. In figure 1 we observe that M_x for the sphere $H = 1$ increases from the initial value 1·06 to the peak value 1·37 at about $a = 0{\cdot}75$, falls to 0·32 at $a = 3{\cdot}50$, then gradually rises to the final value 0·57 as shown by Macagno & Landweber (1960). From (6.1) and (6.4) it can be noted that as the frequency tends to zero, M_x and $M_x^{(2)}$ become non-zero constants while N_x and $N_x^{(2)}$ vanish. For sway, the hydrodynamic quantities decrease in their magnitudes with the reduction of draft. Thus, in the limiting case $H = \infty$ when the body form is a disk, as it is clear from a physical reason, no disturbance will be created by the mode of sway on the free surface.

The variations of added moment of inertia and angular damping coefficient of spheroids and cylinders with rolling frequency are seen in figures 5 and 6, and figures 17 and 18. Since roll and pitch are the same mode for a spheroid, the broken curves in figures 5 and 6 indicate added moment of inertia and associated damping coefficient of a rolling or pitching circular disk. We note that either a flat or a thin body form produces a large wave-making effect in the mode of roll.

a	M_y	M_y^s	N_y	N_y^s	I_x	I_x^s	H_x	H_x^s
For spheroid $H = 2$								
0·50	1·65	1·86	0·88	2·28	0·141	0·206	0·012	0·072
1·00	1·22	1·48	0·79	1·51	0·137	0·147	0·044	0·150
1·50	1·04	1·41	0·63	1·10	0·112	0·113	0·058	0·096
2·00	0·96	1·41	0·49	0·83	0·093	0·094	0·055	0·079
2·50	0·93	1·42	0·38	0·61	0·082	0·084	0·048	0·065
3·00	0·91	1·45	0·24	0·46	0·075	0·077	0·040	0·053
For spheroid $H = 1$								
0·50	1·25	1·46	0·72	1·89	0	0	0	0
1·00	0·91	1·26	0·53	1·00				
1·50	0·82	1·31	0·35	0·60				
2·00	0·81	1·41	0·24	0·34				
2·50	0·83	1·52	0·15	0·20				
3·00	0·87	1·63	0·12	0·14				
For ellipsoid $\bar{b}/\bar{a} = \frac{1}{4}$, $H = 4$								
1·00	0·174	0·121	0·116	0·173	0	0	0	0
1·50	0·138	0·100	0·115	0·142				
2·00	0·113	0·091	0·106	0·118				
2·50	0·097	0·085	0·093	0·099				
3·00	0·087	0·080	0·080	0·084				
For ellipsoid $\bar{b}/\bar{a} = \frac{1}{4}$, $H = 2$					$\times 10^{-1}$	$\times 10^{-1}$	$\times 10^{-1}$	$\times 10^{-1}$
1·00	0·128	0·090	0·094	0·138	0·120	0·137	0·019	0·074
1·50	0·098	0·075	0·084	0·099	0·115	0·090	0·049	0·088
2·00	0·080	0·073	0·068	0·072	0·088	0·071	0·066	0·077
2·50	0·072	0·073	0·052	0·052	0·064	0·063	0·060	0·060
3·00	0·070	0·076	0·039	0·038	0·052	0·058	0·049	0·048

TABLE 1. Strip method approximations

It is clear from (6.1) and (6.4) that the hydrodynamic quantities associated with roll behave in the same fashion as those associated with sway in a lower frequency range. Making use of the cylinder data, the values of the normalized added mass M_y^s, damping coefficient N_y^s for heave, and the normalized added moment of inertia I_x^s and damping coefficient H_x^s for roll, are computed by the strip method formula (4.10). The estimated results are shown together with the actual values in table 1. The strip method does not give a satisfactory estimate for the three-dimensional problem. Beyond the value of $a = 2{\cdot}00$, estimated values of I_x^s and H_x^s approach actual values. For ellipsoids, estimated values of M_y^s and N_y^s also become close to the actual value after passing the value $a = 2{\cdot}00$.

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